Euler’s Polyhedron Formula
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January 27, 2012

Leonhard Euler (1707-1783)
- Contributions to geometry, calculus, number theory, probability, mechanics, astronomy...
- Opera Omnia project started in 1911, includes at least 76 volumes

Euler’s Formula
- “Elementa doctrinae solidorum,” [Elements of the doctrine of solids] 1750
- “Demonstratio nonnullarum insignium proprietatum, quibus solida hedris planis inclusis sunt praedita,” [Proof of some of the properties of solid bodies enclosed by planes.] 1751

Charming Proofs: A Journey into Elegant Mathematics
Claudi Alsina and Roger B. Nelson, MAA Books, 2010

Euler’s Formula
- A polygon is a geometric object “consisting of a number of points (called vertices) and an equal number of line segments (called sides), ...together with the line segments joining consecutive pairs of the points.” (Coxeter and Greitzer 1967, p. 51).

Euler’s Formula
- A polyhedron is a three-dimensional solid which consists of a collection of polygons, joined at their edges.
- A polyhedron is convex if the segment joining two points on the surface of the polyhedron always lies in the polyhedron.
Euler’s Formula

- The polygons that make up a polyhedron are called faces.
- The line segments where two faces meet are called edges.
- The points where three or more faces meet are called vertices (singular: vertex).

Euler’s Formula

- Generally, we let $F$ denote the number of faces, $E$ denote the number of edges, and $V$ denote the number of vertices.
- $F = 6$
- $E = 12$
- $V = 8$

Euler’s Formula

- For any convex polyhedron, $V - E + F = 2$

Euler’s Formula

- According to Ed Sandifer, Euler’s own proof had a flaw.
- Most modern treatments are either too sketchy to be really called proofs, or too dense to easily understand.

“Proof” of Euler’s Formula

- Project the polyhedron to a plane figure so as to preserve the number of faces, edges, and vertices.
Adrien-Marie Legendre (1752-1833)
- Contributions to analysis, number theory, celestial mechanics.
- Translated and adapted Euclid's Elements.

Spherical Geometry
- A form of geometry where "lines" are great circles on the surface of the sphere.

Spherical Geometry
- Three distinct lines form a spherical triangle.

Spherical Geometry
- The angle sum of a triangle in spherical geometry is not 180°. In fact, every triangle has an angle sum of more than 180°.
• It turns out that the area of a triangle is determined by the angle measures.
• In spherical geometry, similar triangles are always congruent.

Suppose that two lines meet at an angle $\theta$.

The lines form two pairs of congruent lunes.
• The area of a pair of lunes is proportional to $\theta$.

$$L(\theta) = \frac{\theta}{\pi} \cdot 4\pi r^2 = 4\theta r^2$$

$\alpha$, $\beta$, $\gamma$
Triangular Area

\[ L(\alpha) = 4\alpha r^2 \]
\[ L(\beta) = 4\beta r^2 \]

Triangular Area

\[ L(\alpha) = 4\alpha r^2 \]
\[ L(\beta) = 4\beta r^2 \]
\[ L(\gamma) = 4\gamma r^2 \]

Polygonal Area

- What about a general polygon? Does its area depend only on its angles?

Polygonal Area

- Let \( P_n \) be a polygon with \( n \) sides.
- Let \( S_\text{sum} \) denote the sum of the measures of the interior angles (in radians)

Polygonal Area

- Triangulate it.
- The area of each triangle is
\[ (\alpha + \beta + \gamma - \pi) r^2 \]
Polygonal Area

\[ A(P_n) = r^2 \left[ S_n - (n-2)\pi \right] \]

Spherical Geometry and Euler’s Formula

- Take any convex polygon, and inscribe it in a sphere of radius 1.

Spherical Geometry and Euler’s Formula

- Project outward to create a spherical polyhedron with the exact same values of F, E, and V.

Spherical Geometry and Euler’s Formula

- The area of the sphere is equal to the sum of the areas of the polygons

\[ 4\pi = \sum A(P_i) \]

Spherical Geometry and Euler’s Formula

- The area of the sphere is equal to the sum of the areas of the polygons

\[ 4\pi = \sum S_i - \sum n_i \pi + \sum 2\pi \]
Spherical Geometry and Euler’s Formula

• The area of the sphere is equal to the sum of the areas of the polygons

\[ 4\pi = \sum S_i - \sum n_i \pi + \sum 2\pi \]

Sum of the measures of the interior angles over all faces

Spherical Geometry and Euler’s Formula

• The area of the sphere is equal to the sum of the areas of the polygons

\[ 4\pi = 2\pi V - \sum n_i \pi + \sum 2\pi \]

Sum of the number of sides of each face over all faces

Spherical Geometry and Euler’s Formula

• The area of the sphere is equal to the sum of the areas of the polygons

\[ 4\pi = 2\pi V - 2\pi E + \sum 2\pi \]

Spherical Geometry and Euler’s Formula

• The area of the sphere is equal to the sum of the areas of the polygons

\[ 4\pi = 2\pi V - 2\pi E + 2\pi F \]
Spherical Geometry and Euler’s Formula

• Divide by $2\pi$ to get Euler’s Formula

$$2 = V - E + F$$

What’s it good for?

• It makes a good exploration project for grade schoolers.
• It can be used to prove that there are exactly five Platonic solids.
• The planar version of the formula can be used to prove the Five Color Theorem.

Theorem

• Any polyhedron has at least one face with five or fewer sides.
• Proof: Suppose for the sake of contradiction that every face has at least six sides.
• By counting edges, we get

$$6F \leq 2E$$

$$F \leq E/3$$

Theorem

• On the other hand, at least three edges meet at each vertex, so

$$3V \leq 2E$$

$$V \leq 2E/3$$

Bibliography

• MacTutor History of Math Archive, http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Euler.html
• Wolfram Mathworld http://mathworld.wolfram.com/
• Proofs and Refutations: The Logic of Mathematical Discovery, Imre Lakatos, Cambridge U Press, 1976